Theorem Let μ be a Radon measure on $\mathbb{R}^d$ such that $0 < \mu(\mathbb{R}^d) < \infty$. Let $F \subset \mathbb{R}^d$ be a Borel set and $0 < c < \infty$ be a constant.

(a) $\forall x \in F, \ \overline{\lim_{r \downarrow 0}} \dfrac{\mu(B(x,r))}{r^s} < c \Rightarrow \mathcal{H}^s(F) \geq \dfrac{\mu(F)}{c}$,

(b) $\forall x \in F, \ \overline{\lim_{r \downarrow 0}} \dfrac{\mu(B(x,r))}{r^s} > c \Rightarrow \mathcal{H}^s(F) \leq 10^s \cdot \dfrac{\mu(\mathbb{R}^d)}{c}$.

Proof (a) For every $\delta > 0$ consider

$$F_\delta := \left\{ x \in F : \frac{\mu(B(x,r))}{r^s} < c, \ \forall 0 < r \leq \delta \right\}.$$

Let $\{U_i\}$ be a δ-cover of F and in this way also a δ-cover of F_δ. If $U_i \cap F_\delta \neq \emptyset$ then we pick $x_i \in U_i \cap F_\delta$. Let $B = B(x_i, |U_i|)$.

Clearly, $U_i \subset B$. Hence,

by the definition of F_δ

$$\mu(U_i) \leq \mu(B) < c|U_i|^s.$$

Hence

$$\mu(F_\delta) \leq \sum_i \{\mu(U_i) : U_i \cap F_\delta \neq \emptyset\} \leq c \cdot \sum_i |U_i|^s$$

Using that $\{U_i\}$ is an arbitrary δ-cover of F:

$$\mu(F_\delta) \leq c \cdot \mathcal{H}^s_\delta(F) \leq c \cdot \mathcal{H}^s(F).$$

Observe that $F_\delta \nearrow F$ as $\delta \downarrow 0$. Hence we get

$\mu(F) \leq c\,\mathcal{H}^s(F)$ as claimed. $\blacksquare$

<u>Proof of (6)</u> We may assume that F is bounded. Fix $\delta > 0$ and let

$$\mathcal{C} := \left\{ B(x,r) : x \in F,\ 0 < r \leq \delta,\ \frac{\mu(B(x,r))}{r^s} > c \right\}.$$

By assumption $F \subset \bigcup_{B \in \mathcal{C}} B$. Using the $5r$ covering theorem $\exists$ $\{B_i\}_i$ disjoint balls

$$B_i \in \mathcal{C}, \quad \bigcup_i 5B_i \supset F.$$

Observe that $\{5B_i\}_i$ is a 10δ-covering of F. Hence,

$$\mathcal{H}^s_{10\delta}(F) \leq \sum_i |5B_i|^s = 5^s \sum_i |B_i|^s \leq 10^s \cdot c^{-1} \sum_i \mu(B_i)$$

$$\leq 10^s c^{-1} \mu(\mathbb{R}^d).$$

Let $\delta \downarrow 0$ to get that $\mathcal{H}^s(F) \leq 10^s c^{-1} \mu(\mathbb{R}^d) < \infty$.

$\blacksquare$

<u>Definition (Local dimension)</u>

Let μ be a Radon measure on $\mathbb{R}^d$. Assume that $x \in \operatorname{spt}(\mu)$. That is we assume that $\mu(B(x,r)) > 0$ for all $r > 0$. Then we define the lower and upper local dimensions of the measure μ as follows:

$$\underline{\dim}_{loc}(\mu,x) := \underline{\lim}_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r}, \quad \overline{\lim}_{loc}(\mu,x) := \overline{\lim}_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r}.$$

If $x \notin spt(\mu)$ then we define

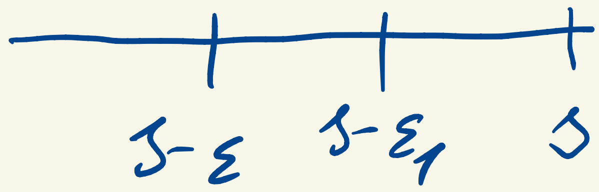
$$\underline{\dim}_{loc}(\mu,x) = \overline{\dim}_{loc}(\mu,x) = \infty.$$

<u>Theorem</u> Let $E \subset \mathbb{R}^d$ be a Borel measure and let μ be a finite Radon measure ($\mu(\mathbb{R}^d) < \infty$). Then

(a) If $\mu(E) > 0$ & $\forall x \in E$ $\underline{\dim}_{loc}(\mu,x) \geq s$ then we have $\dim_H E \geq s$.

(b) If $\forall x \in E$ $\underline{\dim}_{loc}(\mu,x) \leq s$ then $\dim_H E \leq s$.

<u>Proof of (a)</u> Let $0 < \varepsilon_1 < \varepsilon < s$.



If $\underline{\dim}_{loc}(\mu,x) > s-\varepsilon_1$ then $\exists r_0 > 0$ s.t. $\forall 0 < r < r_0$:

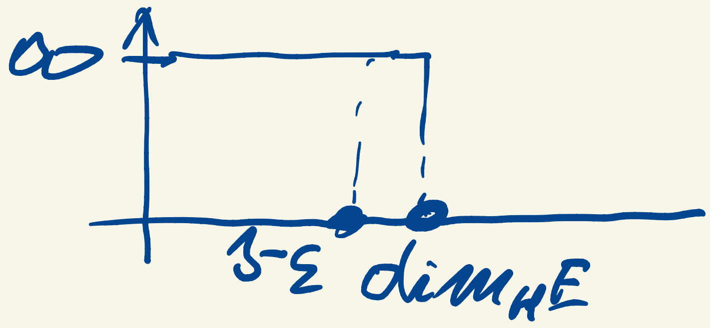
$$\frac{\log \mu(B(x,r))}{\log r} > s-\varepsilon_1. \quad \text{Hence } \mu(B(x,r)) < r^{s-\varepsilon_1} = r^{s-\varepsilon} \cdot r^{\varepsilon-\varepsilon_1}$$

So,
$$\frac{\mu(B(x,r))}{r^{s-\varepsilon}} < r^{\overbrace{\varepsilon-\varepsilon_1}^{>0}} \downarrow 0 \text{ if } r \downarrow 0.$$

Hence $\limsup\limits_{r \downarrow 0} \frac{\mu(B(x,r))}{r^{s-\varepsilon}} < c$, for all $c > 0$.

By the previous theorem this implies that $\mathcal{H}^{s-\varepsilon}(E) > \frac{\mu(E)}{c}$ holds for all $c > 0$. That is

$\mathcal{H}^{s-\varepsilon}(E) = \infty$



$\Downarrow$

$\dim_H E \geq s-\varepsilon$ holds for all $\varepsilon > 0$. That is

$\dim_H E \geq s$. ■

<u>Proof of Part (b)</u> Let $\varepsilon > 0$. Then $\exists\, r_n \downarrow 0$ s.t.

$\frac{\log \mu(B(x,r_n))}{\log r_n} < s+\varepsilon$. That is $\mu(B(x,r_n)) > r_n^{s+\varepsilon}$

Hence $\frac{\mu(B(x,r_n))}{r_n^{s+\varepsilon}} > 1$. That is $\boxed{\limsup\limits_{r \downarrow 0} \frac{\mu(B(x,r_n))}{r^{s+\varepsilon}} > 1 \text{ for } \forall x \in E}$.

This & the previous theorem imply that

$\mathcal{H}^{s+\varepsilon}(E) \leq 10^{s+\varepsilon} \cdot \frac{\mu(\mathbb{R}^d)}{1} \Rightarrow \dim_H E \leq s+\varepsilon$ for all $\varepsilon > 0$

That is $\dim_H E \leq s$. ■

<u>Definition</u> Let μ be a finite Radon measure on $\mathbb{R}^d$. The Hausdorff dimension of μ is

$$\dim_H \mu := \inf\{\dim_H E : \mu(E^c) = 0\}.$$

Some people call this the upper Hausdorff dimension of μ. $\dim_H \mu$ is also used. As opposed to the lower Hausdorff dimension of μ which is

$$\underline{\dim}_H \mu := \inf\{\dim_H E : \mu(E) > 0\}.$$

Theorem $\overline{\dim}_H \mu = \operatorname{ess\,sup}_{\mu} \overline{\dim}_{loc}(\mu, x)$

this means that $\overline{\dim}_{loc}(\mu, x)$ can take values bigger than this only on a set of μ-zero measure of x.

Proof To simplify the notation we write $\alpha := \operatorname{ess\,sup}_{\mu} \overline{\dim}_{loc}(\mu, x)$

Let $E := \{x : \overline{\dim}_{loc}(\mu, x) \leq \alpha\}$. Then $\mu(E^c) = 0$.

by the def of ess sup

Using the previous theorem

$\dim_H E \leq \alpha$. Using this and that we obtain

that $\dim_H \mu \leq \alpha$.

Now we want to prove that $\dim_H \mu \geq \alpha - \varepsilon$ for all $\varepsilon > 0$.

Fix an arbitrary $\varepsilon > 0$. By the definition of the ess sup $\exists E$ with $\mu(E) > 0$ s.t.

$\forall x \in E$, $\underline{\dim}_{loc}(\mu,x) > \alpha - \varepsilon$. Let F be an arbitrary set with $\mu(F^c)=0$. Then $\mu(E \cap F) > 0$ and $\forall x \in E \cap F$: $\underline{\dim}_{loc}(\mu,x) > \alpha-\varepsilon$. That is By the previous theorem we have $\dim_H(E\cap F) \geq \alpha - \varepsilon$. Hence $\dim_H F \geq \alpha - \varepsilon$ whenever $\mu(F^c)=0$. This means that $\dim_H F \geq \alpha$ for all F with $\mu(F^c)=0$. $\blacksquare$

We remark without proof that

$$\underline{\dim}_H(\mu) = \operatorname{ess\,inf} \underline{\dim}_{loc}(\mu,x).$$

Interestingly the lower and upper packing dimension of a measure are

$$\underline{\dim}_p(\mu) := \inf\{\dim_p A : \mu(A)>0\} = \operatorname*{ess\,inf}_{x\sim\mu} \overline{\dim}_{loc}(\mu,x),$$

$$\dim_p(\mu) := \inf\{\dim_p A : \mu(A^c)=0\} = \operatorname*{ess\,sup}_{x\sim\mu} \overline{\dim}_{loc}(\mu,x).$$

Packing dimension of a set $\dim_p A$

Definition We say that a finite or countably infinite collection of disjoint balls $\{B_i\}_i$

is a δ-packing of a set $E \subset \mathbb{R}^d$ if
- the radii of B_i are at most δ and
- the centers of B_i are contained in E

For a $\delta > 0$ we define

$$\mathcal{P}^s_\delta(E) := \sup\{\sum_i |B_i|^s : \{B_i\} \text{ is a } \delta\text{-packing of } E\}.$$

Let

$$\mathcal{P}^s_0(E) := \lim_{\delta \to 0} \mathcal{P}^s_\delta(E).$$

Unfortunately, $\mathcal{P}^s_0$ is NOT countable subadditive so we need one more step:

$$\mathcal{P}^s(E) := \inf\{\sum_{i=1}^{\infty} \mathcal{P}^s_0(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\}.$$

Feng, Hua & Wen proved:

If $E \subset \mathbb{R}^d$ is compact & $\mathcal{P}^s_0(E) < \infty \Rightarrow \mathcal{P}^s_0(E) = \mathcal{P}^s(E)$.

The packing dimension of a set $E \subset \mathbb{R}^d$ is

$$\dim_p E := \inf\{s : \mathcal{P}^s(E) = 0\} = \sup\{s : \mathcal{P}^s(E) = \infty\}.$$

Equivalent definition

$$\boxed{\dim_H E \le \dim_p E \le \overline{\dim}_B E}$$

$$\dim_p E := \inf\{\sup_i \overline{\dim}_B E_i : E \subset \bigcup_{i=1}^{\infty} E_i\}.$$